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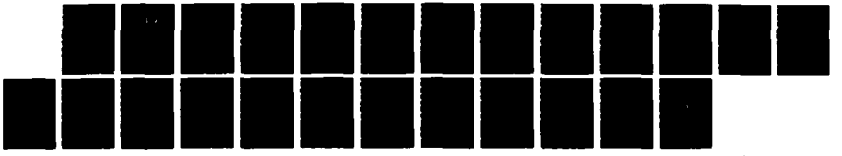

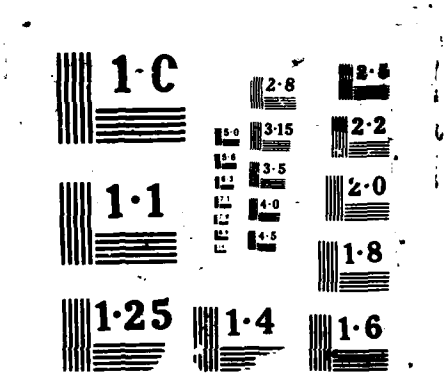
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DE FINETTI'S APPROACH TO GROUP DECISION MAKING

by

Richard E.Barlow*
Sergio Wechsler**
Fabio Spizzichino***

ORC 87-13 June 1987

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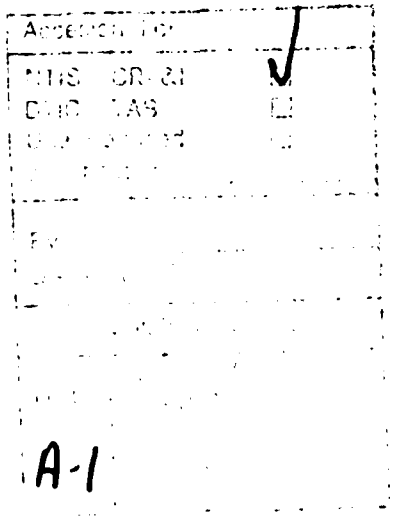
SUMMARY

A group of Bayesians must make a group decision; e.g., choose one of two final actions. De Finetti (1954a,b) considered group decision making, relative to a special sequential decision problem, when all individuals have the same loss function but different opinions. In particular, he defined the "individual horizon" and the "common horizon" relative to a given group decision rule such as a voting rule. He characterized voting rules as an "average of decisions" and argued that it is better for the group to use an "average of opinions." We generalize and extend de Finetti's ideas in his (1954a,b) papers. Finally, we present some of his ideas in his (1959) Varenna lectures.



DE FINETTI'S APPROACH TO GROUP DECISION MAKING

1. INTRODUCTION

A group of individual Bayesians, $I_1, \ldots, I_N$, must make a group decision d_G: i.e., choose one of 2 final actions, a_1 or a_2, relative to an unknown quantity θ. Let ξ_i, $i = 1, 2, \ldots, N$ be the individuals' initial opinions concerning the unknown quantity θ; i.e., these are their probability distributions for θ. The group as a whole has no opinion. Only individuals have opinions. Hence the usual uni-subjective Bayesian approach to decision making does not apply, at least directly. We will only consider the case where individuals in the group have a common interest; i.e., equivalent utility functions and actually identical utility functions in Sections 2 - 5. Throughout the paper we make the strong assumption that the individual prior and utility functions are separable and in fact that priors do not depend on contemplated decisions [cf. Rubin (1987)]. For a recent survey of ideas related to group decision making see French (1984).

Beginning with his 1950 paper "Recent Suggestions for the Reconciliation of Probabilities," de Finetti wrote a series of papers concerning group decision making. These covered at least a ten year period and were based in part on his multisubjective interpretation of Wald's admissibility theorem. In his 1959 Varenna, Italy, lectures, he makes the point that "really new developments in the theory of inductive behaviour [i.e., decision making] arise only in so far as the decisions - unlike the

opinions - are made by groups rather than individuals." [See de Finetti (1972), p. 188.]

In many situations, a group of individuals will make a group decision by voting. A vote is taken and in this way the group "decides." De Finetti interprets certain voting rules as resulting in an "average of decisions" [see Section 2]. However, as de Finetti observes, in certain circumstances this means of reaching a group decision can be improved; but still remaining in the uni-subjective Bayesian framework. De Finetti's main result is that it is better for the group to use an average of opinions rather than an "average of decisions" [de Finetti (1954a)].

The objective of this paper is to extend de Finetti's approach to group decision making, as he presented it, for a class of sequential decision problems, with special attention to the ideas in his 1954 published papers. Section 2 describes the sequential decision problem of interest and the concept of "inductive decision rules" for the group. The setup and results in this section generalize and extend the ideas of de Finetti [(1952), (1954a)].

For a specified loss function $\ell(a_h, \theta)$ (h = 1,2) and a fixed group decision rule, τ (for example a group voting rule), we characterize, in Section 3, the class of Bayes' rules which all individuals in the group would consider as good or better than τ. The "common horizon" characterizes this class of Bayes' rules. This generalizes de Finetti's results in (1954a) and (1954b). Sections 4 and 5 extend de Finetti's results relative to the example described in his (1954a) paper. Section 6 briefly discusses the case when individual utilities are not identical.

2. THE GROUP SEQUENTIAL DECISION PROBLEM

The group must choose one of two final actions (or decisions) a_1 or a_2. Any individual I_i ($i = 1,2,\ldots,N$) in the group has a loss function $\ell_i(a_h,\theta)$ where $\theta \in \Theta$ is the value assumed by an unobservable random quantity θ. $\xi_i^{(o)}$ denotes the probability distribution on Θ representing the initial opinion of I_i about θ and does not depend on the decision contemplated. The group can observe random variables $X_1, X_2, \ldots$ ($X_j \in \mathfrak{X} \subset \Re$, $j = 1,2, \ldots$). In the opinion of all individuals, $X_1, X_2, \ldots$ are conditionally i.i.d. given θ with a given conditional unidimensional density $f(x|\theta)$.

Taking an observation has a constant cost c for any individual in the group. As is usual in the "uni-subjective" framework, it is assumed that, after any observation, the group is allowed either to stop and take one of the two final actions a_1 or a_2 or to take another observation. Denote by a_o the decision to take another observation. $\xi_i^{(n)}$ ($n = 1,2, \ldots$) is the distribution on Θ for the individual I_i at the n-th stage (i.e., after n observations). Obviously $\xi_i^{(n)}$ evolves according to Bayes' formula:

$$d\xi_i^{(n)}(\theta|x_1,\ldots,x_n) \propto f(x_1|\theta)\ldots f(x_n|\theta)\, d\xi_i^{(o)}(\theta)$$

Consider the class of possible sequential decision strategies for the group. A sequential decision strategy is of course defined by a sequence of mappings

$$v^{(n)}: \mathfrak{X}^n \longrightarrow \{a_o, a_1, a_2\} \qquad n = 1,2, \ldots$$

where $v^{(n)}(x_1, \ldots,x_n)$ denotes the decision to be taken by the group at

stage n if the statistical results $x_1, \ldots, x_n$ have been observed. For a fixed vector $\underline{x} \equiv (x_1, \ldots, x_n) \in \mathfrak{X}^n$, $v^{(n)}(x_1, \ldots, x_n)$ is defined only if $v^{(m)}(x_1, \ldots, x_m) \neq a_o$ for all $m < n$.

From a Bayesian point of view the group should take into account only those sequential rules which respect the "likelihood principle" (cf. Berger et al. [1984]); i.e.,

$$f(x_1'|\theta) \ldots f(x_n'|\theta) = f(x_1''|\theta) \ldots f(x_n''|\theta) \quad \forall \ \theta \in \Theta$$
$$\Longrightarrow \ v^{(n)}(\underline{x}') = v^{(n)}(\underline{x}''). \tag{2.1}$$

A remarkable class of such rules is formed by the "inductive rules". An inductive rule can be defined as follows: Let $\mathcal{O}$ be the space of possible probability distributions on Θ ("opinions") and $\mathcal{O}^N = \mathcal{O} \times \mathcal{O} \times \ldots \times \mathcal{O}$ (N times) be the space of possible vectors formed by the opinions of the individuals in the group. Fix a mapping

$$\tau: \ \mathcal{O}^N \longrightarrow \{ a_o, a_1, a_2 \}.$$

At the n-th stage the group takes the action

$$\tau(\underline{\xi}^{(n)}) = \tau(\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_N^{(n)}),$$

where τ does not depend on n. So an inductive rule is characterized by a partition (Ξ_o, Ξ_1, Ξ_2) of the space $\mathcal{O}^N$. A large class of reasonable inductive rules is formed from those which de Finetti called "average of decisions." In order to define an "average of decisions," it is necessary to fix attention on "individual Bayes' rules."

At any stage n (n = 0,1, ...), we denote by $a_i^{*(n)}$ the Bayes decision of individual I_i, depending on the loss function $\ell_i(a_h, \theta)$, his probability distribution $\xi_i^{(n)}$ and the continuation cost c. Obviously, $a_i^{*(n)}$ would be the decision taken by the group at stage n if I_i were a "dictator," who would choose the decision for the group.

Let

$$\varphi: \{a_o, a_1, a_2\}^N \longrightarrow \{a_o, a_1, a_2\}$$

be a "generalized average function" [de Finetti (1931)]; i. e.,

1) φ is symmetric (2.2)

2) if $a_{h_1} = a_{h_2} = \ldots = a_{h_N} = a_h$ $(h = 0,1,2)$ then

$$\varphi(a_{h_1}, a_{h_2}, \ldots, a_{h_n}) = a_h. \tag{2.3}$$

The group decision rule is given by an average of decisions if the group takes, at every stage n, a decision (which does not depend on n)

$$\varphi(a_1^{*(n)}, a_2^{*(n)}, \ldots, a_N^{*(n)}).$$

Particular cases of common interest are obtained by considering "voting rules;" for example de Finetti (1954a):

A) the group stops observations at stage n and chooses a_h $(h = 1,2)$ only if a_i is the value assumed by at least M elements of the set $\{a_1^{*(n)}, \ldots, a_N^{*(n)}\}$. M is a fixed integer between (N+1)/2 (simple majority) and N (unanimity).

B) the group stops observations at stage n and chooses a_h $(h = 1,2)$ only if the number N_h of individuals in the group for which $a_j^{*(n)} = a_h$ is greater than M plus the number N_{3-h} of individuals wanting to stop and take the alternative decision a_{3-h} $(1 \leq M \leq N)$.

C) as in B) but with the additional condition that

$$N_h/(N_{3-h}) > \mu$$

for some $\mu > N/(N-M)$.

We can also consider voting rules with different weights given to different individuals; obviously, in this case, we lose property (2.2).

Individual Bayes' rules corresponding to utilities and initial opinions of single members in the group are very special cases of weighted voting rules. Acting according to a Bayes' rule corresponding to the utility and initial opinion of an individual chosen outside the group is still a rule respecting the likelihood principle, but it is not an average of decisions. For some individual, several Bayes' rules may exist. In this case any randomization between such rules is again a Bayes' rule for the individual. We stress this obvious fact, since randomized Bayes' rules may have a special role in the group decision problem, especially when the distribution of observations is discrete. Note that it is not necessary to specify individual utilities in order to characterize an inductive rule in general. Specification of utilities is, nevertheless, necessary, for the definition of "reasonable" sequential rules for the group.

3. THE "COMMON HORIZON" FOR THE GROUP IN THE CASE OF A COMMON LOSS FUNCTION

Now we fix attention (as de Finetti actually did in [1954a]) on the particular case when

$$\ell_i(a_h,\theta) = \ell(a_h,\theta) \qquad i = 1,2, \ldots, N.$$

Consider a fixed (completely general, in principle) inductive rule chosen by the group, determined by

$$\gamma: \ \mathcal{O}^N \longrightarrow \{a_o, a_1, a_2\}. \tag{3.1}$$

Recall that we denote by $\xi_i^{(o)}$ the i-th individual initial distribution on $\mathcal{B}$.

Think now of a hypothetical individual with the common loss function

$\ell(a_h, \theta)$. Suppose he believes that $\theta = \bar{\theta}$ but must accept, at any stage, the decision chosen by the group. This will result in a loss to him, depending on τ and $\underline{\xi}^{(o)}$, the vector of initial opinions. The expected value of such a loss (as evaluated by the individual himself) will be denoted by

$$\psi_{\bar{\theta}}(\tau, \underline{\xi}^{(o)}) = c\, E(M|\theta=\bar{\theta}) + E[\ell(\tau(\underline{\xi}^{(M)}), \bar{\theta})|\ \theta=\bar{\theta}] \qquad (3.2)$$

where $M = \inf \{m \geq 0|\ \tau(\underline{\xi}^{(m)}) \neq a_o\}$. Obviously $M = 0$, if $\tau(\underline{\xi}^{(o)}) \neq a_o$ in which case the loss to him is deterministic. If the individual is not sure about θ and he assesses a distribution, η, for θ then he has an expected loss

$$\psi_\eta(\tau, \underline{\xi}^{(o)}) = \int \psi_{\bar{\theta}}(\tau, \underline{\xi}^{(o)})\, d\eta(\bar{\theta}). \qquad (3.3)$$

For fixed τ and $\underline{\xi}^{(o)}$, (3.3) obviously defines a linear functional of η.

Consider now an individual with the loss $\ell(a_h, \theta)$ and an initial distribution ξ on Θ. When he is supposed to choose his own strategy, he will choose a Bayes' strategy against ξ [see e.g. De Groot, 1970]. We shall denote by Δ_ξ the set of all Bayes' strategies against ξ (including possibly also the randomized ones). For the generic $\xi \in \mathcal{O}$, Δ_ξ will consist of at least one strategy δ_ξ (and perhaps more if the distribution of observations is discrete). We denote, moreover, by $\rho(\xi)$ the Bayes' risk against ξ and set

$$\Delta = \bigcup_{\xi \in \mathcal{O}} \Delta_\xi.$$

For any $\delta \in \Delta$, we can consider the expected loss $\psi_\eta(\delta)$ of such a strategy as evaluated by the hypothetical individual with the same loss function $\ell(a_h, \theta)$ but with initial distribution η. This is of interest to us, since we want to compare the effects to him of following the group decision

strategy in (3.1) or following the Bayes' strategy of another individual with a different opinion.

Obviously for $\hat{\delta} \in \Delta_\eta$ we have

$$\psi_\eta(\delta) \geq \psi_\eta(\hat{\delta}) = \rho(\eta) \tag{3.4}$$

$$\psi_\eta(\tau, \underline{\xi}^{(o)}) \geq \psi_\eta(\hat{\delta}) = \rho(\eta). \tag{3.5}$$

For fixed $\delta \in \Delta$, $\psi_\eta(\delta)$ is a linear functional of η too. For fixed $\eta \in \mathcal{O}$, $\underline{\xi}^{(o)} \in \mathcal{O}^N$, τ a group inductive rule, we set

$$\Delta_\eta(\tau, \underline{\xi}^{(o)}) = \{ \delta \in \Delta \mid \psi_\eta(\delta) \leq \psi_\eta(\tau, \underline{\xi}^{(o)}) \}.$$

$\delta_{\underline{\xi}} \in \Delta_\eta(\tau, \underline{\xi}^{(o)})$ means that the individual with initial opinion η prefers to act according to the Bayes' strategy of a fellow with initial opinion $\underline{\xi}$, rather than according to the strategy τ of the group, where the i-th single individual's opinion is represented by $\xi_i^{(o)}$ $(i = 1, 2, \ldots, N)$.

$\Delta_\eta(\tau, \underline{\xi}^{(o)})$ is not empty since, obviously,

$$\Delta_\eta(\tau, \underline{\xi}^{(o)}) \supseteq \Delta_\eta$$

by (3.5). The sets $\Delta_\eta(\tau, \underline{\xi}^{(o)})$, $\eta \in \mathcal{O}$ are the Bayesian "individual horizons" with respect to τ, in the terminology introduced by de Finetti. With reference to a fixed inductive rule τ, the "common horizon for the group" is the set of (possibly randomized) individual Bayes' rules defined by the intersections of individual horizons of the members of the group:

$$\Delta_{\underline{\xi}^{(o)}}(\tau) = \bigcap_{i=1}^{N} \Delta_{\xi_i^{(o)}}(\tau, \underline{\xi}^{(o)}).$$

$\tau' \in \Delta_{\underline{\xi}^{(o)}}(\tau)$ is a decision rule which is Bayes for some individual and whose initial opinion may possibly differ from $\xi_1^{(o)}, \ldots, \xi_N^{(o)}$. For any member of the group, τ' is preferable to τ. In [1954a], de Finetti shows, by studying in detail a particular example, that in the case when $\otimes$ has

only two values, it is possible to obtain explicitly individual horizons and common horizons for a group, by means of elementary geometry. We shall illustrate this in the next section.

4. THE CASE OF A PARAMETER SPACE WITH ONLY TWO VALUES

As mentioned, the Bayesian individual horizons and the Bayesian common horizon can be constructed by elementary geometric tools in the case when

$$\Theta = \{\theta_1, \theta_2\}. \tag{4.1}$$

In the following, a probability distribution on Θ will be represented by the quantity

$$\xi = P\{\theta = \theta_2\}. \tag{4.2}$$

It is no restriction to assume $\theta_1, \theta_2 \in \mathcal{R}$ and $\theta_1 < \theta_2$. By (4.2) we can let $\mathcal{O}$ coincide with $[0,1]$ and stochastic ordering among elements of $\mathcal{O}$ will coincide with the natural linear ordering for the real numbers: for P', $P'' \in \mathcal{O}$

$$P' \underset{st}{\leq} P''$$

$$\Longleftrightarrow \quad \xi' \equiv P'\{\theta = \theta_2\} \leq \xi'' \equiv P''\{\theta = \theta_2\}.$$

It is convenient to label individuals in the group so that

$$\xi_1^{(o)} \leq \xi_2^{(o)} \leq \ldots \leq \xi_N^{(o)}. \tag{4.3}$$

We shall also label a_1 and a_2 in such a way that

$$\ell(a_h, \theta_k) = 0 \qquad \text{if } h = k \quad (h,k = 1,2)$$

$$\ell(a_h, \theta_k) > 0 \qquad \text{if } h \neq k.$$

Fix now an arbitrary inductive rule τ and compute the two values

$$\psi_{\theta_1}(\tau, \underline{\xi}^{(o)}), \quad \psi_{\theta_2}(\tau, \underline{\xi}^{(o)}).$$

According to the definition (3.2) for $\psi_{\bar{\theta}}(\tau, \underline{\xi}^{(o)})$ $(\bar{\theta} \in \Theta)$, $\psi_{\theta_1}(\tau, \underline{\xi}^{(o)})$

is the expected loss for a hypothetical individual $\hat{I}$ characterized by the following situation:

(a) $\hat{I}$ has a loss function $\ell(a_h, \theta_k)$

(b) $\hat{I}$ must accept the decision strategy τ chosen by the group of individuals $I_1, \ldots, I_N$ with initial opinions $\underline{\xi}^{(o)}$

(c) $\hat{I}$ is from the beginning sure that $\theta = \theta_1$ and

$$\eta = P\{\theta = \theta_2\} = 0$$

for him.

$\psi_{\theta_2}(\tau, \underline{\xi}^{(o)})$ has an analogous meaning.

In the Cartesian plane, the segment joining the two points with coordinates

$$(0, \psi_{\theta_1}(\tau, \underline{\xi}^{(o)})), \quad (1, \psi_{\theta_2}(\tau, \underline{\xi}^{(o)})$$

is the graph of the linear function

$$W_\tau(\eta) = \psi_\eta(\tau, \underline{\xi}^{(o)}) \qquad (0 \leq \eta \leq 1). \tag{4.4}$$

$W_\tau(\eta)$ is the expected loss for a hypothetical individual $\hat{I}$ with the situation described by (a), (b) and (c'), where

(c') $\hat{I}$ has initial opinion η about θ; i.e.,

$$P\{\theta = \theta_2\} = \eta.$$

As in the last section, we denote by $\psi_\eta(\delta)$, the expected loss of a Bayes' strategy $\delta \in \Delta$ for an individual with initial opinon η. The graph of $\psi_\eta(\delta)$ versus η is again a segment on a straight line. We consider the family of all such segments, together with the graph of the function $\rho(\eta)$, representing the risk associated with the Bayes' strategies as a function of the initial opinion η $(0 \leq \eta \leq 1)$. $\rho(\eta)$ is the curve formed by points in the plane with coordinates $(\eta, \psi_\eta(\delta_\eta))$. By (3.4) the graph of $\psi_\eta(\delta_\xi)$ is tangent to $\rho(\eta)$ at $\eta = \xi$.

Some well-known properties for $\rho(\eta)$ follow immediately. In particular

$\rho(\eta) = \eta\ell(a_1, \theta_2)$ in some right neighborhood of $\eta = 0$

$\rho(\eta) = (1-\eta)\ell(a_2, \theta_1)$ in some left neighborhood of $\eta = 1$

$\rho(\eta) \leq \min(\eta\ell(a_1, \theta_2), (1-\eta)\ell(a_2, \theta_1))$, all $\eta \in [0,1]$

$\rho(\eta)$ is concave.

Moreover, $\rho(\eta)$ is a continuous piecewise linear function when the statistical observations $x_1, x_2, \ldots$ have a discrete distribution since, in such a case, a Bayes' sequential decision strategy with respect to a given initial opinion will be Bayes also with respect to all initial opinions lying in some interval containing that opinion.

Now we fix an inductive rule τ and we want to determine the corresponding individual horizon for an individual with initial opinion η. Consider in the plane, the point $Q_{\eta,\tau}$ with coordinates $(\eta, W_\tau(\eta))$. It follows that

$$W_\tau(\eta) \geq \rho(\eta).$$

We are interested in the case $W_\tau(\eta) > \rho(\eta)$. From $Q_{\eta,\tau}$ we draw the two tangent lines to the curve ρ and denote by $\xi_1(\eta,\tau)$ and $\xi_2(\eta,\tau)$, the abscissas of the two contact points (see Figure 4.1). From elementary considerations, it is now possible to prove the following result: The individual horizon with respect to τ is formed by the Bayes' rules corresponding to opinions ξ such that

$$\xi_1(\eta,\tau) \leq \xi \leq \xi_2(\eta,\tau).$$

If $\xi_k(\eta,\tau)$ $(k = 1,2)$ is a corner point for ρ then one must consider only those Bayes' strategies corresponding to lines tangent to ρ at $\xi_k(\eta,\tau)$ and passing through points of the kind (η,y) with $y \leq W_\tau(\eta)$.

The common horizon with respect to τ for the group of individuals with initial opinions $\xi^{(o)}$ is formed by Bayes' rules corresponding to opinions ξ such that

$$\xi_1(\xi_N^{(o)}, \tau) \leq \xi \leq \xi_2(\xi_1^{(o)}, \tau).$$

If $\xi_1(\xi_N^{(o)}, \tau) = \xi_2(\xi_1^{(o)}, \tau)$ and this is a corner point, then the common horizon may be formed only by randomized Bayes' rules. We remark that $\xi_1(\eta, \tau)$ and $\xi_2(\eta, \tau)$ are corner points $\forall\ \eta \in \mathcal{O}$ if the observations have a discrete distribution.

It may happen in general that the strategies in the common horizon are Bayes' only with respect to initial opinions different from those of the individuals in the group. It is also possible that they could be randomized and not correspond to even a single hypothetical opinion.

5. VOTING RULES AND GROUP SEQUENTIAL DECISION PROBLEMS

The Case $\Theta = \{\theta_1, \theta_2\}$

In the uni-subjective case with $\Theta = \{\theta_1, \theta_2\}$, the Bayes' strategies are the so-called Wald sequential probability ratio tests [cf. Ferguson (1967), pp. 361-368]. Depending on the loss function, $\ell(a_h, \theta_k)$ and the cost of sampling, c, there exist numbers $\xi_L < \xi_U$ such that if

$$\xi_L < \xi_i^{(n)} < \xi_U$$

the Bayes' decision rule for individual I_i at stage n is to continue sampling and to stop and take decision a_1 (a_2) if

$$\xi_i^{(n)} \leq \xi_L$$
$$(\xi_i^{(n)} \geq \xi_U).$$

It is easy to show that

$$\xi_1^{(o)} \leq \xi_2^{(o)} \leq \ldots \leq \xi_N^{(o)} \tag{5.1}$$

implies

$$\xi_1^{(n)} \leq \xi_2^{(n)} \ldots \leq \xi_N^{(n)} \tag{5.2}$$

for n = 1,2, ... Hence an M out of N voting group decision rule will be determined by individuals I_M and I_{N-M+1}. Obviously, if N is odd and M = (N+1)/2, the simple majority rule, then the median individual determines the simple majority group decision rule. In this case, the simple majority rule is also a Bayes' decision rule.

The Case When Θ is an Interval of the Real Line

The uni-subjective case of the sequential two terminal action decision problem for $f(x|\theta)$ a member of the exponential family of the form

$$dF(x|\theta) = \psi(\theta)\, e^{x\theta}\, d\mu(x)$$

was solved by Sobel [1953]. In this case (y(n),n) is a sufficient statistic for θ where, given observations $x_1, x_2, \ldots, x_n$

$$y(n) = \sum_1^n x_i.$$

We suppose that the loss function $\ell(a_h, \theta)$ (h = 1,2) has the following properties: $\ell(a_1, \theta)$ is increasing (that is, non-decreasing) in θ; $\ell(a_2, \theta)$ is decreasing (that is, non-increasing) in θ. The cost of another observation at stage n is $c_n > 0$.

At stage n of sequential sampling, the Bayes' rule with respect to a density (or discrete probability) $\xi(\theta)$ on Θ can be characterized by two numbers, namely $y_1(n) < y_2(n)$. The Bayes' rule is

1) Continue sampling if $y_1(n) < y(n) < y_2(n)$;

2) Stop and take action a_1 if $y(n) \leq y_1(n)$;

3) Stop and take action a_2 if $y(n) \geq y_2(n)$.

We can generalize (5.1) and (5.2) for this case as follows. Assume that $\xi_i(\theta)$ is totally positive of order 2 in $i = 1, 2, \ldots, N$ and $\theta \in \Theta$ (i. e., is TP_2), that is, there is an ordering of individuals in the group such that the following 2x2 determinant is non-negative when $i < j$ and for all $\theta_1 < \theta_2$ belonging to Θ:

$$\begin{vmatrix} \xi_i(\theta_1) & \xi_i(\theta_2) \\ \xi_j(\theta_1) & \xi_j(\theta_2) \end{vmatrix} \geq 0 \qquad (5.3)$$

It is easy to see that when

$$\Theta = \{\theta_1, \theta_2\}$$

(5.3) reduces to (5.1). (Recall that $\xi_i(\theta_2) = 1 - \xi_i(\theta_1)$ in this case.)

Since $\xi_i(\theta|y(n))$ is proportional to $p(y(n)|\theta)\ \xi_i(\theta)$, it is obvious that $\xi_i(\theta|y(n))$ is also TP_2 in $i = 1, 2, \ldots, N$ and $\theta \in \Theta$. This generalizes (5.2).

Example. If $\xi_i(\theta) = \theta^{A_i}(1 - \theta)^{B_i}$ and

$$0 \leq A_1 \leq A_2 \leq \cdots \leq A_N$$

$$B_1 \geq B_2 \geq \cdots \geq B_N \geq 0,$$

then (5.3) is satisfied

Let $[y_{i1}(n), y_{i2}(n)]$ be the Bayes' rule corresponding to individual I_i at the n-th stage of sampling. By definition

$$\int_{\Theta} \ell(a_1, \theta)\ \xi_i(\theta|y_{i1}(n))\ d\theta < \int_{\Theta} \ell(a_2, \theta)\ \xi_i(\theta|y_{i1}(n))\ d\theta,$$

that is, individual I_i would prefer action a_1 to action a_2 if

$$y(n) = y_{i1}(n)$$

at stage n.

Lemma 5.1. For $h < i$,

$$\int_{\Theta} \ell(a_1, \theta)\, \xi_h(\theta|y_{i1}(n))\, d\theta \;<\; \int_{\Theta} \ell(a_2, \theta)\, \xi_h(\theta|y_{i1}(n))\, d\theta. \quad (5.4)$$

For $i < j$

$$\int_{\Theta} \ell(a_1, \theta)\, \xi_j(\theta|y_{i2}(n))\, d\theta \;>\; \int_{\Theta} \ell(a_2, \theta)\, \xi_j(\theta|y_{i2}(n))\, d\theta. \quad (5.5)$$

Proof

For $h < i$, it follows from the sign variation diminishing theorem [Karlin (1968)] that

$$\int \ell(a_1, \theta)\, \xi_h(\theta|y_{i1}(n))\, d\theta \;\leq\; \int \ell(a_1, \theta)\, \xi_i(\theta|y_{i1}(n))\, d\theta$$

$$< \int_{\Theta} \ell(a_2, \theta)\, \xi_i(\theta|y_{i1}(n))\, d\theta \;\leq\; \int_{\Theta} \ell(a_2, \theta)\, \xi_h(\theta|y_{i1}(n))\, d\theta$$

since $\ell(a_1, \theta)$ is increasing in θ while $\ell(a_2, \theta)$ is decreasing in θ. For $h < i$ and $y(n) = y_{i1}(n)$, it follows that I_h would prefer a_1 to a_2 even more than would I_i.

(5.5) follows in a similar way. QED

Theorem 5.1 Let $\xi_i(\theta)$ be TP_2 in i and θ. If at the n-th stage $y(n) \leq y_{i1}(n)$, then individuals $h \leq i$ would prefer either to continue observation or to stop and take action a_1 rather than to stop and take action a_2.

Similarly, if at the n-th stage $y(n) \geq y_{i2}(n)$, then individuals $j > i$ would either prefer to continue observation or to stop and take action a_2 rather than to stop and take action a_1.

Proof. From inequality (5.4) in Lemma 1 it follows that if $h < i$, then

$$y_{h2}(n) \geq y_{i1}(n)$$

Hence $y(n) \leq y_{i1}(n)$ implies that $h < i$ would prefer either to continue observation or to stop and take action a_1 rather than to stop and take action a_2.

The proof is similar for $j > i$ and $y(n) \geq y_{i2}(n)$. QED

M-Out-Of-N Voting Rules

Following an M-out-of-N voting rule, the group stops and takes action a_h $(h = 1,2)$ as soon as M or more members are in favor of stopping and taking action a_h. From Lemma 5.1 we can show that

$$y_{h,2}(n) \geq y_{i,1}(n) \qquad \text{when } h < i$$

and

$$y_{j,1}(n) \leq y_{i,2}(n) \qquad \text{when } i < j.$$

Hence, if at the n-th stage, $y(n) \geq y_{M,2}(n)$, then the group will either stop and take action a_2 or continue to take observations, since there will be less than M in favor of stopping and taking action a_1.

If, at the n-th stage, $y(n) \leq y_{N-M+1}$ then the group will either stop and take action a_1 or continue to take observations, since there will be less than M in favor of stopping and taking action a_2.

If N is odd and $M = (N+1)/2$, then individual I_M will not necessarily determine the final group action to be taken as in the case $\Theta = (\theta_1, \theta_2)$.

6. DETERMINING THE GROUP DECISION RULE

In his summer 1959, Varenna, Italy, lectures [cf. de Finetti (1972), pp. 196-196], de Finetti discusses the general group decision problem. In the last of 10 points, he makes the following controversial statement: "Greater complications are encountered with more widely differing attitudes

and interests of the individuals. But no new criterion is called for: one has but to apply the criterion of the maximumal expected utility in different circumstances." He also presents several examples illustrating how a group of individuals, whose utilities are based on money, might reach an initial group decision rule.

Individuals With Shares in a Joint Economic Enterprise

Suppose that group losses are distributed according to share. If individual i has share w_i, decision d is taken and θ occurs, then i loses $w_i L(d,\theta)$. Individual I_i will prefer to use d_i, his best decision rule with his opinion ξ_i. The sum of the loss shares is the "group loss" $L(d,\theta)$

There is no possible agreement between the group members as long as each considers his individual uni-subjective decision problem - even if they have the same shared loss function. Each individual has his own Bayes' rule with his opinion. The individuals MUST negotiate if the group is to make a decision. There seem to be two levels of thinking here, the individual level and the group level. The initial decision is reached by each individual considering his own decision problem. A compromise is necessary for the sake of reaching a group decision.

De Finetti suggests a compromise that leads to a group decision rule entirely acceptable to each individual. He suggests a reallocation of the group loss: individual i bears the total loss if his own Bayes' rule is actually used. Under this convention, we can consider the "randomized" group rule which selects d_i with probability w_i. The expected loss to individual i is then precisely his expected share of the group's loss were the group to agree to use his Bayes' rule with probability one (and thus

minimize his true expected loss). Therefore, every individual would be satisfied "as much as if he alone were to make the decision which for him is optimal" [de Finetti (1972), p. 196].

However, the group as a whole will have even a lower expected loss, in EVERYONE'S opinion, if the (possibly randomized) decision rule corresponding to the parallel tangent to the risk curve is used. (Notice that the former compromised rule is not, in general, Bayesian against any possible opinion - not even of a hypothetical individual opinion). By using the rule corresponding to the parallel tangent, the amount of improvement is the same in everyone's opinion. This is the result that de Finetti considers better for all - meaning that each would consider the expected GROUP loss reduced. However, in I_i's opinion, this will be WORSE for him.

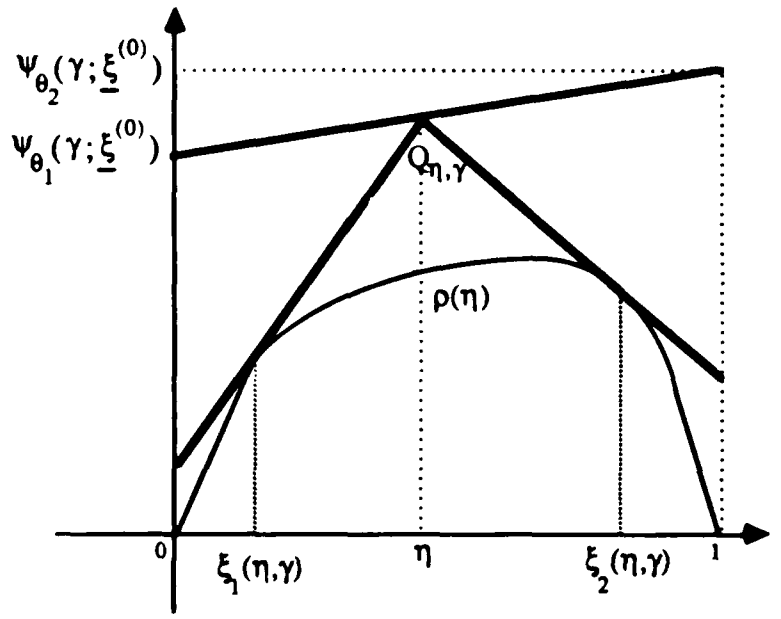


Figure 4.1
The Individual Horizon

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